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OPEN

DOI 10.1515/tmj-2017-0024

Domain of the generalized double Cesàro matrix in some paranormed spaces of double sequences

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Abstract

In this paper, we investigate some double paranormed sequence spaces which are domain of the double generalized Cesàro matrix R^q in some double sequence spaces. We also determined of $\alpha-$dual of $\mathcal{R}_u(t)$ which is the space of double sequences whose R^q-transforms are bounded and examined some linear and topological properties of these sequence spaces.

2010 Mathematics Subject Classification. **40B05**. 46A45, 40C05
Keywords. Double sequence spaces, four dimensional Riesz matrix of order one, paranormed double sequence spaces.

1 Introduction

We denote the sets of all real or complex valued double sequences by Ω. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$ then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p - \lim x_{mn} = l$. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the set of all such sequences by $\mathcal{C}_r$.

Throughout the paper $t = (t_{mn})$ will denote a double sequence of strictly positive numbers. The following paranormed sequence space were examined by Gökhan and Çolak [3, 4, 5]

$$\begin{aligned}
\mathcal{M}_u(t) &:= \left\{(x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty\right\},\\
\mathcal{C}_p(t) &:= \left\{(x_{mn}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 0\right\},\\
\mathcal{C}_{0p}(t) &:= \left\{(x_{mn}) \in \Omega : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 0\right\},\\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t).
\end{aligned}$$

When all terms are of $t = (t_{mn})$ are constant and all equal to positive number t we have the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$ of double sequences of are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both bounded and convergent in the Pringsheim's sense and both bounded and null in the Pringsheim's sense, respectively. Móricz [1] proved that $\mathcal{C}_{bp}, \mathcal{C}_{bp0}, \mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|.\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}|$.

Tbilisi Mathematical Journal 10(2) (2017), pp. 43–56.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 26 October 2016.
Accepted for publication: 23 November 2016.

Let A denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn-th term to Ax as follows:

$$(Ax)_{m,n} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{mnkl}x_{kl}.$$

The $v-$*summability domain* $\lambda_A^{(v)}$ of a four dimensional infinite matrix $A = (a_{mnkl})$ in a space λ of a double sequences is defined by

$$\lambda_A^{(v)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(v - \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}. \tag{1}$$

Now, we can mention certain studies related to the double sequence spaces. In her PhD thesis, Zeltser [2] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [6] have introduced the statistical convergence and statistical Cauchy for double sequences, and gave the relation between statistically convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [7] and Mursaleen and Edely [8] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. Altay and Başar [9] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the alpha-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(v)-$duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. More recently, Başar and Sever [10] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of absolutely $q-$summable single sequences and examine some properties of the space $\mathcal{L}_q$. Demiriz and Duyar [12] have defined the spaces $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ of double sequences whose difference transforms are bounded , convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and absolutely $q-$summable, respectivel. Also they have examined some inclusion relations related to these sequence spaces and determined the alpha-dual of the space $\mathcal{M}_u(\Delta)$ and the $\beta(v)-$dual of the space $\mathcal{C}_\eta(\Delta)$ and characterized the some matrix classes.

Let $(q_k), (s_l)$ be sequences of non-negative numbers which are not all zero and

$$Q_m = \sum_{k=0}^{m} q_k; \quad q_0 > 0 \qquad \text{and} \qquad S_n = \sum_{l=0}^{n} s_l; \quad s_0 > 0. \tag{2}$$

Then the four-dimensional generalized Cesàro or Riesz matrix $R^q = (r^q_{mnkl})$ is defined by

$$r^q_{mnkl} := \begin{cases} \dfrac{q_k s_l}{Q_m S_n} & , \quad 0 \le k \le m, \quad 0 \le l \le n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m,n,k,l \in \mathbb{N}$ [15]. In the present paper, we assume that the terms of the double sequences $x=(x_{mn})$ and $y=(y_{mn})$ are connected with the relation

$$y_{mn}=(R^q x)_{mn}=\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl} \tag{3}$$

for all $m,n\in\mathbb{N}$ which is called the Riesz transform of $x=(x_{mn})$. If we take $q_k=t_l=1$ for all $k\in\{0,1,...m\}$ and $l\in\{0,1,...n\}$ then we obtain the four-dimensional Cesàro matrix $C=(c_{mnkl})$ of order one is defined by

$$c_{mnkl}:=\begin{cases}\dfrac{1}{(m+1)(n+1)} & , \quad 0\le k\le m, \quad 0\le l\le n,\\ 0 & , \quad \text{otherwise}\end{cases}$$

for all $m,n,k,l\in\mathbb{N}$. The Cesàro transform of a double sequence $x=(x_{mn})$ is defined by

$$(Cx)_{mn}=\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij} \tag{4}$$

for all $m,n\in\mathbb{N}$.

Mursaleen and Başar [11] have defined the spaces $\widetilde{\mathcal{M}}_u$, $\widetilde{\mathcal{C}}_p$, $\widetilde{\mathcal{C}}_{0p}$, $\widetilde{\mathcal{C}}_{bp}$, $\widetilde{\mathcal{C}}_r$ and $\widetilde{\mathcal{L}}_q$ of double sequences whose Cesàro mean transforms are bounded , convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and Cesàro absolutely $q-$summable, respectively, that is,

$$\begin{aligned}
\widetilde{\mathcal{M}}_u &:= \left\{(x_{ij})\in\Omega : \sup_{m,n\in\mathbb{N}}\frac{1}{(m+1)(n+1)}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|<\infty\right\},\\
\widetilde{\mathcal{C}}_p &:= \left\{(x_{ij})\in\Omega : \exists\, l\in\mathbb{C}\ni p-\lim_{m,n\to\infty}\frac{1}{(m+1)(n+1)}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}-l\right|=0\right\},\\
\widetilde{\mathcal{C}}_{0p} &:= \left\{(x_{ij})\in\Omega : p-\lim_{m,n\to\infty}\frac{1}{(m+1)(n+1)}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|=0\right\},\\
\widetilde{\mathcal{L}}_u &:= \left\{(x_{ij})\in\Omega : \sum_{m,n}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|<\infty\right\}.
\end{aligned}$$

Furthermore, they have studied some topological properties of these spaces and characterized some matrix classes.

Recently, Demiriz and Duyar [13] have defined and studied the double paranormed Cesàro sequence spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t)$, $\widetilde{\mathcal{C}}_{0p}(t)$ and $\widetilde{\mathcal{L}}_u(t)$, that is,

$$\begin{aligned}
\widetilde{\mathcal{M}}_u(t) &:= \left\{(x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left|\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} < \infty\right\}, \\
\widetilde{\mathcal{C}}_p(t) &:= \left\{(x_{mn}) \in \Omega : \exists\, l \in \mathbb{C} \ni p-\lim_{m,n\to\infty} \left|\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} - l\right|^{t_{mn}} = 0\right\}, \\
\widetilde{\mathcal{C}}_{0p}(t) &:= \left\{(x_{mn}) \in \Omega : p-\lim_{m,n\to\infty} \left|\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} = 0\right\}, \\
\widetilde{\mathcal{L}}_u(t) &:= \left\{(x_{ij}) \in \Omega : \sum_{m,n} \left|\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} < \infty\right\}.
\end{aligned}$$

and the spaces $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{C}}_r(t)$ the sets of all the paranormed Cesàro convergent and bounded, and the paranormed Cesàro regularly convergent double sequences. When all terms of (t_{mn}) are constant and all are equal to $t > 0$, then we obtain $\widetilde{\mathcal{M}}_u(t) = \widetilde{\mathcal{M}}_u$, $\widetilde{\mathcal{L}}_u(t) = \widetilde{\mathcal{L}}_q$ and when all terms of (t_{mn}), excluding the first finite number of m and n, are constant and all are equal to $t > 0$, then we obtain $\widetilde{\mathcal{C}}_p(t) = \widetilde{\mathcal{C}}_p$ and $\widetilde{\mathcal{C}}_{0p}(t) = \widetilde{\mathcal{C}}_{0p}$, see [11]. One can easily see that the spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t)$, $\widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_r(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$ are the domain of the double Cesàro matrix C in the spaces $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{C}_r(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{L}_u(t)$, respectively.

2 Some new paranormed spaces of double sequences

In the present section, we introduce the new double paranormed Riesz sequence spaces $\mathcal{R}_u(t)$, $\mathcal{R}_p(t)$, $\mathcal{R}_{0p}(t)$, $\mathcal{R}^{\mathcal{L}}(t)$ and we give some topological and algebraical properties of these spaces. We define the new double paranormed Riesz sequence spaces $\mathcal{R}_u(t)$, $\mathcal{R}_p(t)$, $\mathcal{R}_{0p}(t)$, $\mathcal{R}^{\mathcal{L}}(t)$ as following,

$$\begin{aligned}
\mathcal{R}_u(t) &:= \left\{(x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left|\frac{1}{Q_m S_n} \sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}} < \infty\right\}, \\
\mathcal{R}_p(t) &:= \left\{(x_{mn}) \in \Omega : \exists\, l \in \mathbb{C} \ni p-\lim_{m,n\to\infty} \left|\frac{1}{Q_m S_n} \sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl} - l\right|^{t_{mn}} = 0\right\}, \\
\mathcal{R}_{0p}(t) &:= \left\{(x_{mn}) \in \Omega : p-\lim_{m,n\to\infty} \left|\frac{1}{Q_m S_n} \sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}} = 0\right\}, \\
\mathcal{R}^{\mathcal{L}}(t) &:= \left\{(x_{mn}) \in \Omega : \sum_{m,n} \left|\frac{1}{Q_m S_n} \sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}} < \infty\right\}.
\end{aligned}$$

Also, by $\mathcal{R}_{bp}(t)$ and $\mathcal{R}_r(t)$, we denote the sets of all the paranormed Riesz convergent and bounded, and the paranormed Riesz regularly convergent double sequences. Actually we can redefine the new sequence spaces $\mathcal{R}_u(t)$, $\mathcal{R}_p(t)$, $\mathcal{R}_{0p}(t)$, $\mathcal{R}^{\mathcal{L}}(t)$, $\mathcal{R}_{bp}(t)$ and $\mathcal{R}_r(t)$ are the domain of the double Riesz matrix R^q in the spaces $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_r(t)$, respectively. In addition, when all terms of (t_{mn}) are constant and all are equal to $t > 0$, then we obtain $\mathcal{R}_u(t) = \mathcal{R}^{qt}(\mathcal{M}_u)$, $\mathcal{R}^{\mathcal{L}}(t) = \mathcal{R}^{qt}(\mathcal{L}_s)$ and when all terms of (t_{mn}), excluding the first finite number of m and n, are constant and all are equal to $t > 0$, then we obtain $\mathcal{R}_p(t) = \mathcal{R}^{qt}(\mathcal{C}_p)$ and $\mathcal{R}_{0p}(t) = \mathcal{R}^{qt}(\mathcal{C}_{0p})$, see [14].

Theorem 2.1. The space $\mathcal{R}_u(t)$ is a linear space if and only if $H = \sup_{m,n} t_{mn} < \infty$.

Proof. First we deal with the sufficiency. Take $x, y \in \mathcal{R}_u(t)$ firstly, and suppose that $H < \infty$ and $M = \max\{1, H\}$. Then, there exist some K_x and K_y such that

$$\left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}} \leq K_x^M \quad \text{and} \quad \left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l y_{kl}\right|^{t_{mn}} \leq K_y^M$$

for all $m, n \in \mathbb{N}$. Since $t_{mn}/M < 1$ for all $m, n \in \mathbb{N}$, one can see that

$$\begin{aligned}\left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l (x_{kl} + y_{kl})\right|^{t_{mn}/M} &\leq \left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}/M} \\ &+ \left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l y_{kl}\right|^{t_{mn}/M} \\ &\leq K_x + K_y.\end{aligned}$$

Hence we have that $x + y \in \mathcal{R}_u(t)$.

Secondly, take $x \in \mathcal{R}_u(t)$ and $\lambda \in \mathbb{C}$. Then since the inequality

$$|\lambda|^{t_{mn}} \leq \max\{1, |\lambda|^M\}$$

holds for all $m, n \in \mathbb{N}$, it is easily obtain that

$$\begin{aligned}\left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l \lambda x_{kl}\right|^{t_{mn}} &= |\lambda|^{t_{mn}}\left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}} \\ &\leq \max\{1, |\lambda|^M\}\left|\frac{1}{Q_m S_n}\sum_{k=0}^{m}\sum_{l=0}^{n} q_k s_l x_{kl}\right|^{t_{mn}}\end{aligned}$$

which is show that $\lambda x \in \mathcal{R}_u(t)$.

Now consider the necessity. Suppose that $\mathcal{R}_u(t)$ be a linear space but $H = \sup_{m,n} t_{mn} = \infty$. Then, there exist the sequences $(m(i))$ and $(n(j))$, one of them is strictly increasing and the other one is non-decreasing, such that

$$t_{m(i),n(j)} > i + j \tag{5}$$

for all positive integers i, j. Now, we consider the sequence

$$x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2} b^{m(i)n(j)}$$

where $b^{mn} = (b_{ij}^{mn})_{ij}$ defined by

$$b_{ij}^{mn} = \begin{cases} \dfrac{Q_m S_n}{q_m s_n}, & i = m, \quad j = n, \\[2ex] -\dfrac{Q_m S_n}{q_{m+1} s_n}, & i = m+1, \quad j = n, \\[2ex] -\dfrac{Q_m S_n}{q_m s_{n+1}}, & i = m, \quad j = n+1, \\[2ex] \dfrac{Q_m S_n}{q_{m+1} s_{n+1}}, & i = m+1, \quad j = n+1, \\[2ex] 0, & \text{otherwise} \end{cases} \tag{6}$$

for all $m, n \in \mathbb{N}$. Then, we have the sequence $y = (y_{mn})$ where

$$y_{mn} = \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} = \begin{cases} \dfrac{1}{2}, & m = m(i), \quad n = n(j), \\[2ex] 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Therefore, it follows by (5) that

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}} &= \sup_{i,j\in\mathbb{N}} |y_{m(i),n(j)}|^{t_{m(i),n(j)}} \\ &= \sup_{i,j\in\mathbb{N}} 2^{-t_{m(i),n(j)}} \\ &\leq \sup_{i,j\in\mathbb{N}} 2^{-i-j} \leq \frac{1}{2}. \end{aligned}$$

Hence, $x \in \mathcal{R}_u(t)$ but it is clearly that $4x \notin \mathcal{R}_u(t)$ which contradicts the fact that $\mathcal{R}_u(t)$ is a linear space. Therefore, we have that $H < \infty$.

Theorem 2.2. $\mathcal{R}_{bp}(t)$ and $\mathcal{R}^{\mathcal{L}}(t)$ are linear spaces if and only if $H = \sup_{m,n} t_{mn} < \infty$.

Proof. The proof of this Theorem is similar to Theorem 2.1. So, we omit the detail.

Theorem 2.3. The space $\mathcal{R}_p(t)$ is a linear space if and only if $T = \lim_{N\to\infty} \sup_{m,n\geq N} t_{mn} < \infty$.

Proof. Take $x, y \in \mathcal{R}_p(t)$ and suppose that $T < \infty$. Then there exists some complex numbers L_1, L_2 such that

$$p - \lim_{m,n\to\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} - L_1 \right|^{t_{mn}} = 0$$

and

$$p - \lim_{m,n\to\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j y_{ij} - L_2 \right|^{t_{mn}} = 0$$

also there exists a $\tau > 0$ such that $t_{mn} < \tau$ for all sufficiently large m, n. Then, consider the following inequality

$$\left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_j(x_{ij}+y_{ij})-(L_1+L_2)\right|^{t_{mn}/\tau} \leq \left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_jx_{ij}-L_1\right|^{t_{mn}/\tau} + \left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_jy_{ij}-L_2\right|^{t_{mn}/\tau}$$

we have that $x+y \in \mathcal{R}_p(t)$. Furthermore, let $\gamma \in \mathbb{C}$ and $x \in \mathcal{R}_p(t)$. Then, by the inequality

$$\left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_j\gamma x_{ij}-\gamma L_1\right|^{t_{mn}} \leq \max\{1,|\gamma|^{\tau}\}\left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_jx_{ij}-L_1\right|^{t_{mn}}$$

we obtain $\gamma x \in \mathcal{R}_p(t)$.

Conversely, let $\mathcal{R}_p(t)$ be a linear space and suppose that $T = \infty$. Then, there exists a strictly increasing sequence $(N(i,j))$ of positive integers such that $t_{m(i),n(j)} > i+j$ for all $i,j \in \mathbb{N}$, where $m(i), n(j) \geq N(i,j) > 1$. So, we consider the sequence

$$x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2}b^{m(i)n(j)}.$$

Then, it is clear that $x = (x_{mn}) \in \mathcal{R}_p(t)$ but since

$$|y_{m(i),n(j)}| = \left|\frac{1}{Q_{m(i)+1}S_{n(j)+1}}\sum_{i=0}^{m(i)}\sum_{j=0}^{n(j)}4q_is_jx_{ij}\right|^{t_{m(i),n(j)}} = 2^{t_{m(i),n(j)}} > 2^{i+j}$$

$4x \notin \mathcal{R}_p(t)$. Hence T must be finite.

Theorem 2.4. The space $\mathcal{R}_{0p}(t)$ is a linear space if and only if $T = \lim_{N\to\infty}\sup_{m,n\geq N} t_{mn} < \infty$.

Proof. The proof of this Theorem is similar to Theorem 2.3. So, we omit the detail.

Theorem 2.5. Let $H = \sup_{m,n} t_{mn}$ and $M = \max\{1, H\}$. Then, the space $\mathcal{R}_u(t)$ is a complete paranormed space with g defined by

$$g(x) = \sup_{m,n\in\mathbb{N}}\left|\frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_jx_{ij}\right|^{t_{mn}/M}$$

if and only if $h = \inf_{m,n} t_{mn} > 0$.

Proof. Suppose that $\mathcal{R}_u(t)$ be a paranormed space with the paranorm g but $h = 0$. Consider the sequence $(x^l) = (x_{ij}) \subset \mathcal{R}_u(t)$ $(l \in \mathbb{N})$ defined by $x_{ij} = 1$ for all $i,j \in \mathbb{N}$ and the sequence $\gamma = (\gamma_l) = (1/(1+l))$ of scalars such that

$$\gamma_l \to 0 \quad \text{as} \quad l \to \infty \quad \text{and} \quad \frac{1}{Q_mS_n}\sum_{i=0}^{m}\sum_{j=0}^{n}q_is_jx_{ij} = 1$$

for all $m, n \in \mathbb{N}$. Since

$$g(\gamma_l x^l) = \sup_{m,n\in\mathbb{N}} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j \gamma_l x_{ij}^l \right|^{t_{mn}/M} = \sup |\gamma_l|^{t_{mn}/M} = 1,$$

we obtain a contradiction with $g(\gamma_l x^l) \to 0$ as $l \to \infty$. Therefore, $h > 0$.

Conversely, let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$ for all $x \in \mathcal{R}_u(t)$. Also, it is clear that $g(x + y) \leq g(x) + g(y)$ for all $x, y \in \mathcal{R}_u(t)$.

Moreover, let (x^l) be any sequence in $\mathcal{R}_u(t)$ such that $g(x^l - x) \to 0$ $(l \to \infty)$ and (γ_l) also be any sequence of scalars such that $\gamma_l \to \gamma$ $(l \to \infty)$. Then, there exists a positive real number K such that $|\gamma_l| \leq K$ for all $l \in \mathbb{N}$. Thus, we have

$$\begin{aligned} g(\gamma_l x^l - \gamma x) &= \sup_{m,n\in\mathbb{N}} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j (\gamma_l x_{ij}^l - \gamma x_{ij}) \right|^{t_{mn}/M} \\ &\leq \sup_{m,n\in\mathbb{N}} \left(|\gamma_l| \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^l - \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right| \right)^{t_{mn}/M} \\ &+ \sup_{m,n\in\mathbb{N}} \left(|\gamma_l - \gamma| \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j \right| \right)^{t_{mn}/M} \\ &\leq K g(x^l - x) + |\gamma_l - \gamma|^{t_{mn}/M} g(x). \end{aligned}$$

Therefore, we obtain that $g(\gamma_l x^l - \gamma x) \to 0$ as $l \to \infty$. So, we show that $(\mathcal{R}_u(t), g)$ is a paranormed space.

For the proof of completeness of the space $(\mathcal{R}_u(t), g)$ suppose that (x^r) be any Cauchy sequence in the space $\mathcal{R}_u(t)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$g(x^r - x^s) = \sup_{m,n\in\mathbb{N}} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j (x_{ij}^r - x_{ij}^s) \right|^{t_{mn}/M} < \varepsilon$$

for all $r, s > N$. So, we have that

$$\begin{aligned} &\left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^r - \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^s \right|^{t_{mn}/M} \\ \leq\ & \sup \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^r - \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^s \right|^{t_{mn}/M} < \varepsilon, \end{aligned}$$

which shows that

$$\left(\frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^r \right)_{r\in\mathbb{N}}$$

is a Cauchy sequence of complex numbers. Then, this sequence converges from completeness of $\mathbb{C}$, say

$$\left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^r - \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^s \right|^{t_{mn}/M} \to 0 \quad \text{as} \quad r \to \infty.$$

We now show that $x \in \mathcal{R}_u(t)$. Since (x^r) is a Cauchy sequence in the space $\widetilde{\mathcal{M}}_u(t)$, there exists a positive numbers K such that $g(x^r) < K$. Also, if we consider the following inequality

$$\begin{aligned} g(x) &= \sup_{m,n\in\mathbb{N}} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}/M} \\ &\leq g(x^r) + \sup_{m,n\in\mathbb{N}} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} - \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x^r_{ij} \right|^{t_{mn}/M} \end{aligned}$$

then we have that by passing to limit as $r \to \infty$,

$$g(x) \leq K + \varepsilon$$

which leads us to the fact that $x \in \mathcal{R}_u(t)$.

Theorem 2.6. Let $N_1 = \min\{n_0 : \sup_{m,n\geq n_0} |y_{mn}|^{t_{mn}} < \infty\}$, $N_2 = \min\{n_0 : \sup_{m,n\geq n_0} t_{mn} < \infty\}$ and $N = \max\{N_1, N_2\}$. Then, the spaces $\mathcal{R}_p(t)$ and $\mathcal{R}_{0p}(t)$ is a complete paranormed spaces with the paranorm g defined by

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}/M}$$

if and only if $\mu > 0$, where $\mu = \lim_{N\to\infty} \inf_{m,n\geq N} t_{mn}$ and $M = \max\{1, \sup_{m,n\geq N} t_{mn}\}$.

Proof. The proof of this Theorem is similar to the proof of Theorem 2.5. So, we omit the detail.

Theorem 2.7. The space $\mathcal{R}^{\mathcal{L}}(t)$ is a complete paranormed space with the paranorm g defined by

$$g(x) = \left[\sum_{m,n=1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} \right]^{1/M}$$

where $M = \max\{1, H\}$ and $H = \sup_{m,n} t_{mn} < \infty$.

Proof. It is clear that $g(\theta) = 0$, $g(-x) = g(x)$. Let $x, y \in \mathcal{R}^{\mathcal{L}}(t)$, then we have that

$$\begin{aligned} g(x) &\leq \left[\sum_{m,n=1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} \right]^{1/M} + \left[\sum_{m,n=1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j y_{ij} \right|^{t_{mn}} \right]^{1/M} \\ &\leq g(x) + g(y). \end{aligned}$$

Furthermore, for any $\lambda \in \mathbb{C}$, we have $g(\lambda x) \leq \max\{1, |\lambda|\} g(x)$. Hence, $(\lambda, x) \to \lambda x$ is continuous at $\lambda = 0$, $x = \theta$, and that the function $x \to \lambda x$ is continuous at $x = \theta$ whenever λ is a fixed scalar. If $x \in \mathcal{R}^{\mathcal{L}}(t)$ is fixed, and $\varepsilon > 0$, we can choose $M, N > 1$ such that

$$\begin{aligned} R(x) &= \sum_{m=1}^{M} \sum_{n=N+1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} + \sum_{m=M+1}^{\infty} \sum_{n=1}^{N} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} \\ &+ \sum_{m=M+1}^{\infty} \sum_{n=N+1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} < \varepsilon/2. \end{aligned}$$

Therefore, $R(\lambda x) < \varepsilon/2$ since $|\lambda| < 1$ and $\delta > 0$, so that $|\lambda| < \delta$ gives

$$\sum_{m,n=1}^{M,N} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j \lambda x_{ij} \right|^{t_{mn}} < \varepsilon/2.$$

Thus, $|\lambda| < \min\{1, \delta\}$ implies that $g(\lambda x) < \varepsilon$. Hence, the function $\lambda \to \lambda x$ is continuous at $\lambda = 0$. So, $\mathcal{R}^{\mathcal{L}}(t)$ is a paranormed space.

Now, we show that the space $(\mathcal{R}^{\mathcal{L}}(t), g)$ is complete. Let (x^r) be any Cauchy sequence in the space $\mathcal{R}^{\mathcal{L}}(t)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$\begin{aligned} g(x^r - x^s) &= \left[\sum_{m,n=1}^{\infty} \left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j (x_{ij}^r - x_{ij}^s) \right|^{t_{mn}} \right]^{1/M} \quad (7) \\ &= \left[\sum_{m,n=1}^{\infty} |y_{mn}^r - y_{mn}^s|^{t_{mn}} \right]^{1/M} < \varepsilon \end{aligned}$$

for all $r, s > N$. Thus, we have that

$$|y_{mn}^r - y_{mn}^s| \leq g(x^r - x^s) < \varepsilon$$

which show that

$$y^r = (y_{mn}^r)_{r\in\mathbb{N}} = \left(\frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij}^r \right)_{r\in\mathbb{N}}$$

is a Cauchy sequence of complex numbers for every fixed $m, n \in \mathbb{N}$. Then, this sequence converges from completeness of $\mathbb{C}$, say (y_{mn}^r) is convergent to y_{mn}, namely,

$$\lim_{r\to\infty} y_{mn}^r = y_{mn} \quad (8)$$

for all $m, n \in \mathbb{N}$. So we can define the sequence $y = (y_{mn})$. Furthermore, one can see by (7) that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |y_{mn}^r - y_{mn}^s|^{t_{mn}} < \varepsilon^M$$

for $r, s > N$. If we pass to the limit as $s \to \infty$, then we have that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |y_{mn}^r - y_{mn}|^{t_{mn}} < \varepsilon^M$$

for $r > N$ by (8). Thus, the last inequality lead us to the fact that $g(x^r - x) \leq \varepsilon$ for $r > N$, which gives the required result.

Theorem 2.8. The double sequence spaces $\mathcal{R}_u(t)$, $\mathcal{R}_p(t)$,, $\mathcal{R}_{0p}(t)$, $\mathcal{R}^{\mathcal{L}}(t)$, $\mathcal{R}_{bp}(t)$ and $\mathcal{R}_r(t)$ are linearly isomorphic to the spaces $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_r(t)$, respectively, where $0 < t_{mn} \leq H$.

Proof. Let consider the space $\mathcal{R}_u(t)$. We can define a transformation $T : \mathcal{R}_u(t) \to \mathcal{M}_u(t)$ by the equality (3). It is clear that T is linear. Further, it is obvious that T is injective since $x = \theta$ whenever $Tx = \theta$.

Now, suppose that $y = (y_{mn}) \in \mathcal{M}_u(t)$ and define the sequence $x = (x_{mn})$ by

$$x_{mn} = \frac{1}{q_m s_n}\left\{Q_m S_n y_{mn} - Q_{m-1} S_n y_{m-1,n} - Q_m S_{n-1} y_{m,n-1} + Q_{m-1} S_{n-1} y_{m-1,n-1}\right\}$$

for all $m, n \in \mathbb{N}$. Then, we have that

$$g(x) = \sup_{m,n\in\mathbb{N}} \left|\frac{1}{Q_m S_n}\sum_{i=0}^{m}\sum_{j=0}^{n} q_i s_j x_{ij}\right|^{t_{mn}/M} = \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}/M} < \infty.$$

Hence, the last inequality show that $x \in \mathcal{R}_u(t)$ and so T is surjective and is paranorm preserving. Thus, the spaces $\mathcal{R}_u(t)$ and $\mathcal{M}_u(t)$ are are linearly isomorphic.

Theorem 2.9. The $\alpha-$dual of the space $\mathcal{R}_u(t)$ for every $t = (t_{mn})$ is the set $M_2^{\infty}(t)$ which is defined by

$$M_2^{\infty}(t) = \bigcap_{N>1}\left\{a = (a_{mn}) \in \Omega : \sum_{k,l} |a_{mn}| N^{1/t_{mn}} < \infty\right\}.$$

Proof. Suppose that $a = (a_{mn}) \in M_2^{\infty}(t)$ and $x = (x_{mn}) \in \mathcal{R}_u(t)$. Then, there exists a positive integer N such that

$$|y_{mn}|^{t_{mn}} = \left|\frac{1}{Q_m S_n}\sum_{i=0}^{m}\sum_{j=0}^{n} q_i s_j x_{ij}\right|^{t_{mn}} \le \max\left\{1, \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}}\right\} < N$$

for all $m, n \in \mathbb{N}$. Thus we have the following inequality

$$\begin{aligned}
\sum_{m,n} |a_{mn} x_{mn}| &= \sum_{m,n}\left|a_{mn}\frac{1}{q_m s_n}\sum_{k=m-1}^{m}\sum_{l=n-1}^{n}(-1)^{m+n-(k+l)} Q_k S_l y_{kl}\right| \\
&\le \sum_{m,n} |a_{mn}|\frac{1}{q_m s_n} N^{1/tmn}\left|\sum_{k=m-1}^{m}\sum_{l=n-1}^{n}(-1)^{m+n-(k+l)} Q_k S_l\right| \\
&= \sum_{m,n} |a_{mn}| N^{1/tmn}.
\end{aligned}$$

Hence we have that $a \in \{\mathcal{R}_u(t)\}^{\alpha}$, i.e., $M_2^{\infty}(t) \subset \{\mathcal{R}_u(t)\}^{\alpha}$.

Otherwise, $a \in \{\mathcal{R}_u(t)\}^{\alpha}$ but suppose that $a \notin M_2^{\infty}(t)$. Then, there exists a positive integer $N > 1$ such that $\sum_{m,n} |a_{mn}| N^{1/tmn} = \infty$. Now we consider the $x = (x_{mn})$ defined by

$$x_{mn} = N^{1/tmn}\text{sgn}a_{mn}$$

for all $m, n \in \mathbb{N}$. Then, one can easily show that $x \in \mathcal{R}_u(t)$, but we have $a \notin \{\mathcal{R}_u(t)\}^{\alpha}$ since

$$\sum_{m,n} |a_{mn} x_{mn}| = \sum_{m,n} |a_{mn}| N^{1/tmn} = \infty.$$

This contradicts show that $a \in M_2^{\infty}(t)$, i.e, $\{\mathcal{R}_u(t)\}^{\alpha} \subset M_2^{\infty}(t)$.

Theorem 2.10. (i) $\mathcal{R}_u(t) \subset \mathcal{R}^{qt}(\mathcal{M}_u)$ if and only if $h = \inf t_{mn} > 0$.
(ii) $\mathcal{R}^{qt}(\mathcal{M}_u) \subset \mathcal{R}_u(t)$ if and only if $H = \sup t_{mn} < \infty$.
(iii) $\mathcal{R}_u(t) = \mathcal{R}^{qt}(\mathcal{M}_u)$ if and only if $0 < h \leq H < \infty$.

Proof. (i) Let $\mathcal{R}_u(t) \subset \mathcal{R}^{qt}(\mathcal{M}_u)$ but $h = 0$. Then, there exist the sequences $(m(i))$ and $(n(j))$, one of them is strictly increasing and the other one is non-decreasing, such that

$$t_{m(i),n(j)} < \frac{1}{i+1} \tag{9}$$

for all positive integers i, j. Let us define the sequence

$$x = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} i b^{m_i n_j}.$$

Then, we have the sequence $y = (y_{mn})$ where

$$y_{mn} = \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} = \begin{cases} i, & m = m(i), \quad n = n(j), \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}} &= \sup_{i,j\in\mathbb{N}} |y_{m(i),n(j)}|^{t_{m(i),n(j)}} \\ &= \sup_{i,j\in\mathbb{N}} i^{t_{m(i),n(j)}} \\ &\leq \sup_{i,j\in\mathbb{N}} i^{\frac{1}{1+i}} \leq 2. \end{aligned}$$

Therefore $x \in \mathcal{R}_u(t)$ but it is clearly that $x \notin \mathcal{R}^{qt}(\mathcal{M}_u)$ which is a contradiction. Hence, $h > 0$.

Conversely, let $x \in \mathcal{R}_u(t)$ and $h > 0$. Then, there exists a positive real number K_x such that

$$\left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right|^{t_{mn}} \leq K_x,$$

for all $m, n \in \mathbb{N}$. So, we have that

$$\left| \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} \right| \leq K_x^{1/t_{mn}} \leq \max\{1, K_x^{1/t}\},$$

which leads us to the consequence that $x \in \mathcal{R}^{qt}(\mathcal{M}_u)$.

(ii) Suppose that $\mathcal{R}^{qt}(\mathcal{M}_u) \subset \mathcal{R}_u(t)$ but $H = \infty$. Then there exist the sequences $(m(i))$ and $(n(j))$, one of them is strictly increasing and the other one is nondecreasing, such that

$$t_{m(i),n(j)} > i + j$$

for all $i, j \in \mathbb{N}$. In this case if we consider the sequence $x = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 2b^{m_i n_j}$ for which

$$y_{mn} = \frac{1}{Q_m S_n} \sum_{i=0}^{m} \sum_{j=0}^{n} q_i s_j x_{ij} = \begin{cases} 2, & m = m(i), \quad n = n(j), \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Then, we obtain that $x \in \mathcal{R}^{qt}(\mathcal{M}_u)$ but $x \notin \mathcal{R}_u(t)$ since $|y_{mn}|^{t_{mn}} = 2^{t_{m(i),n(j)}} > 2^{i+j}$, which is a contradiction, i.e., $H < 0$.

Conversely, let $x \in \mathcal{R}^{qt}(\mathcal{M}_u)$ and $H < 0$. Then, there exists a positive real number K_x such that $|y_{mn}| \leq K_x$ for all $m, n \in \mathbb{N}$. Hence we have that

$$|y_{mn}|^{t_{mn}} \leq K_x^{t_{mn}} \leq \max\{1, K_x^H\}.$$

Therefore, $x \in \mathcal{R}_u(t)$.

(iii) Follows from (i) and (ii).

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